

Uniform Versions of Infinitary Properties in Banach Spaces

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Abstract

In functional analysis it is of interest to study the following general question:

Is the uniform version of a property that holds in all Banach spaces also valid in all Banach spaces?

Examples of affirmative answers to the above question are the host of proofs of almost-isometric versions of well known isometric theorems. Another example is Rosenthal's uniform version of Krivine's Theorem. Using an extended version of Henson's Compactness result for positive bounded formulas in normed structures, we show that the answer of the above question is in fact yes for every property that can be expressed in a particular infinitary language. Examples of applications are given.

1 Introduction

A natural type of questions in functional analysis asks if the “almost” version of a theorem true in a class of normed spaces is also true in the class. Here

are some examples of such questions:

1. Ulam's Theorem:

- **Theorem 1.1.** *Let $T : X \rightarrow Y$ be an onto function from a Banach space X to a Banach space Y with $T(0) = 0$ such that:*

$$\|T(x) - T(y)\| = \|x - y\| \text{ for all } x, y \in X$$

then $\|T(x + y) - T(x) - T(y)\| = 0$ for $x, y \in X$

- Gevirtz ([3]) proved the following "almost" version:

Theorem 1.2. *Let $T : X \rightarrow Y$ be an onto function from a Banach space X to a Banach space Y with $T(0) = 0$ such that $\forall x, y \in X$:*

$$(1 - \epsilon)\|x - y\| \leq \|T(x) - T(y)\| \leq (1 + \epsilon)\|x - y\|$$

then $\|T(x + y) - T(x) - T(y)\| \leq \epsilon'(\|x\| + \|y\|)$ for $x, y \in X$, where $\epsilon' \rightarrow 0$ as $\epsilon \rightarrow 0$.

2. A classical result of Behrends ([1]):

- A linear projection $P : E \rightarrow E$ is called an L^p -projection, $1 \leq p \leq \infty$, if $\forall x \in E$,

$$(\|P(x)\|^p + \|x - P(x)\|^p)^{1/p} = \|x\|,$$

with the obvious modification for the case $p = \infty$.

Behrends proved the following:

Theorem 1.3. *Let E be a Banach space with $\dim(E) > 2$. Let $1 \leq p, q \leq \infty$, $p \neq 2$ and such that $P, Q : E \rightarrow E$ are L^p and L^q projections. Then $p = q$ and $\|PQ - QP\| = 0$.*

- The "almost" isometric case was proved by Cambern, Jarosz and Wodinski ([2]):

Theorem 1.4. *Let E be a Banach space with $\dim(E) > 2$. Let $1 \leq p, q \leq \infty$, $p \neq 2$ and let $P, Q : E \rightarrow E$ be projections with the additional properties:*

$$\forall x \in E, (1 - \epsilon)\|x\| \leq \|P(x)\|^p + \|x - P(x)\|^p)^{1/p} \leq (1 + \epsilon)\|x\|$$

and

$$\forall x \in E, (1 - \epsilon)\|x\| \leq \|Q(x)\|^q + \|x - Q(x)\|^q)^{1/q} \leq (1 + \epsilon)\|x\|$$

then

$$|p - q| \leq \epsilon'(p) \text{ and } \|PQ - QP\| \leq \epsilon'(p), \text{ where } \epsilon' \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Jarosz ([9]) pointed out that the proof of the two above results could be simplified considerably by using ultraproducts of Banach spaces. One may ask then if it is possible to study this phenomena in a systematic way from a logical point of view.

The natural model theoretic setting to answer this question is Henson's logic of positive bounded formulas in normed spaces ([4]). This logic L_{PB} is closed under finite conjunction, finite disjunction and bounded quantification. The normed spaces that are the natural models for this language are called normed space structures. Henson defined the notion of an n -approximation of a formula ϕ in L_{PB} , denoted by $(\phi)_n$. From this concept he defined the semantic notion of approximate truth (\models_{AP}) for this logic. It can be seen that (L_{PB}, \models_{AP}) has a compactness theorem (see [4] and [5]).

However, L_{PB} has a fundamental limitation for our purposes: formulas of the form $\phi \Rightarrow \psi$ (like Ulam's Theorem or Behrend's result), with ϕ, ψ

in L_{PB} , are not in L_{PB} . Furthermore, since most interesting statements in functional analysis are of a fully infinitary type (using countable disjunction for example), we are interested in obtaining a general uniformity result that includes infinitary formulas with countable disjunctions and conjunctions and bounded quantification over infinitely many variables.

We deal with this limitation by extending the notion of approximate truth (\models_{AP}) to an infinitary logic L_A that contains L_{PB} and is closed under countable conjunctions (\bigwedge), negation (\neg) and bounded existential quantification over countably many variables ($\exists \vec{x}(\bigwedge_{i=1}^{\infty} ||x_i|| \leq r_i \wedge \dots)$). This logic and its corresponding notion of \models_{AP} was introduced in [12] to study the idea of “proof by approximation” in analysis from a logical point of view.

Let us describe briefly how we extend the notion of approximation of a formula from L_{PB} to L_A . The main obstacle to the extension of approximate truth to L_A is the negation connective. The key to solving this is to extend Henson’s idea of a sequence of approximate formulas $\{(\phi)_n : n \in \omega\}$ (for formulas $\phi \in L_{PB}$) to a tree of positive bounded formulas $\{([\phi]_h)_n : h \in I(\phi), n \in \omega\}$ where each sequence $([\phi]_h)_1, ([\phi]_h)_2, \dots, ([\phi]_h)_n, \dots$ is a branch of the tree. In this way a sentence ϕ is approximately true in a normed structure E iff there exists a branch of the tree of approximations of ϕ such that all the approximations of this branch hold in E . An analogue to this approach in classical infinitary logic is the notion of approximation of infinitary formulas by Vaught sentences ([6]).

Since the approximations for the infinitary formulas in L_A are in turn positive bounded formulas, we can invoke Henson’s theorem for L_{PB} to obtain a compactness result for (L_A, \models_{AP}) . From this compactness theorem we get the following general uniformity result for formulas of the form $\neg\phi$:

Uniformity Theorem for L_A .

For any class of normed space structures axiomatized by a theory Σ in L_{PB} , for any sentence $\phi \in L_A$, if $\Sigma \models \neg\phi$ then for every branch $h \in I(\phi)$ there exists an integer n such that $\Sigma \models \neg([\phi]_h)_n$.

This paper is organized as follows: For the sake of completeness, the first two sections are devoted to a brief review of Henson's notion of a normed space structure as well as the definition of the logic L_A (Section 2) and a review of the definition of \models_{AP} in L_A (Section 3).

In Section 4 we use Henson's Compactness theorem for L_{PB} to get a Model Existence Theorem for (L_A, \models_{AP}) . From this theorem we prove the Uniformity Theorem.

Finally, in Section 5 we give applications of the Uniformity Theorem to Banach space theory. Among the applications we can cite the theorems of Gervitz and Jarosz mentioned above, as well as Rosenthal's uniform version of Krivine's Theorem.

A note on notation: we will use \blacksquare to denote the end of definitions, examples and remarks.

2 Normed space structures and the logic L_A

We begin by briefly recalling Henson's notions of normed space structure and of a language for normed space structures. For a more detailed account, the reader may look at [4] or [7].

Definition 2.1. Real valued m-ary relations.

A real valued m-ary relation on a normed space E is a function $\mathcal{R}: E^m \rightarrow \mathbb{R}$ which is uniformly continuous on every bounded subset of E^m . \blacksquare

Definition 2.2. A normed space structure is a structure of the form

$$\mathbf{E} = (E, f_i, \mathcal{R}_j : i \in I, j \in J)$$

with:

- E being a normed space structure over the reals;
- each f_i being a function $f_i : E^m \rightarrow E$ for some natural number m ;
- every f_i being a uniformly continuous function on every bounded subset of E^m ;
- each \mathcal{R}_j being a real valued relation. ■

From the concept of normed space structure follows the notion of a signature for normed space structures. A signature in this setting consists of function symbols, real valued relation symbols, bounds for the function symbols and the real valued relation symbols. It also contains moduli of uniform continuity for the function and real valued relation symbols on each bounded set.

Notation. We will use $||\cdot||$ for norms, and $|\cdot|$ for absolute value.

Definition 2.3. A signature for the normed space structure

$$\mathbf{E} = (E, f_i, \mathcal{R}_j : i \in I, j \in J)$$

consists of:

1. an m -ary function symbol for each m -ary function f_i ;
2. an m -ary relation symbol for each m -ary real valued relation \mathcal{R}_i ;

3. for each m -ary function f_i , each positive integer N and each positive rational ϵ , a positive rational $\delta(f_i, N, \epsilon)$ such that

$$\begin{aligned} \|x_k\|, \|y_k\| < N \text{ and } \|x_k - y_k\| < \delta(f_i, N, \epsilon) \ (1 \leq k \leq m) \\ \text{implies } \|f_i(x_1, \dots, x_m) - f_i(y_1, \dots, y_m)\| < \epsilon; \end{aligned}$$

4. for each m -ary real valued function \mathcal{R}_j , each positive integer N and each $V \in \mathcal{V}$, a positive rational $\delta(\mathcal{R}_j, N, V)$ such that

$$\begin{aligned} \|x_k\|, \|y_k\| < N \text{ and } \|x_k - y_k\| < \delta(\mathcal{R}_j, N, V) \ (1 \leq k \leq m) \\ \text{implies } (\mathcal{R}_j(x_1, \dots, x_m) - \mathcal{R}_j(y_1, \dots, y_m)) \in V; \end{aligned}$$

5. for each m -ary function f_i , for each integer N , an integer $K(i, N)$ such that

$$\|x_k\| \leq N \ (1 \leq k \leq m) \text{ implies } \|f(\vec{x})\| \leq K(i, N);$$

6. for each m -ary relation \mathcal{R}_j , for each integer N , an integer $K(j, N)$ such that:

$$\|x_k\| \leq N \ (1 \leq k \leq m) \text{ implies } |\mathcal{R}_j(\vec{x})| \leq K(j, N). \blacksquare$$

If Ω is a signature for the normed space structure \mathbf{E} , we say that \mathbf{E} is an Ω -structure.

Remark 2.4. A constant in this signature is a 0-ary function. \blacksquare

For every signature for normed structures, a first order language is associated in the following way.

Definition 2.5. Let Ω be a signature for a normed space structure with universe E . We associate to it the following first order language consisting of:

- a constant symbol 0, a binary function symbol +, and for each rational scalar r , a function symbol for the scalar multiplication $x \rightarrow rx$;
- for each rational number r , predicate symbols for the sets

$$\{x \in E : \|x\| \leq r\} \text{ and } \{x \in E : \|x\| \geq r\}$$

- the function symbols of Ω ;
- for each real valued relation symbol \mathcal{R} in Ω , for each rational r , predicate symbols for the sets:

$$\{(x_1, \dots, x_m) \in E^m : \mathcal{R}(x_1, \dots, x_m) \leq r\} \text{ and } \\ \{(x_1, \dots, x_m) \in E^m : \mathcal{R}(x_1, \dots, x_m) \geq r\}. \blacksquare$$

The formulas of L_{PB} are defined by induction. As usual, for every formula ϕ , we will use the notation $\phi(\vec{x})$ to express the fact that the free variables of ϕ are among the components of the vector \vec{x} . Likewise, $\phi(\vec{x}_1, \vec{x}_2, \dots)$ means that the free variables of ϕ are among the components of the vectors $\vec{x}_1, \vec{x}_2, \dots$.

Definition 2.6. Definition of L_{PB} .

Fix a signature Ω .

1. If t is a term of the first order language corresponding to Ω and r is a rational number, then $\|t\| \leq r$ and $\|t\| \geq r$ are formulas in L_{PB} .
2. Let \mathcal{R} be an m -ary real valued relation, t_1, \dots, t_m be terms and r be a rational number, then $\mathcal{R}(t_1, \dots, t_m) \leq r$ and $\mathcal{R}(t_1, \dots, t_m) \geq r$ are formulas in L_A .
3. If ϕ_1, ϕ_2 are formulas in L_{PB} , then $\phi_1 \wedge \phi_2$ is a formula in L_{PB} .

4. if $\{\phi_i\}_{i=1}^\infty$ is a countable collection of formulas in L_{PB} , then $\bigwedge_{i=1}^\infty \phi_i \in L_{PB}$.
5. If ϕ_1, ϕ_2 are formulas in L_{PB} then $\phi_1 \vee \phi_2$ is a formula in L_{PB} .
6. Consider a formula $\phi(y, \vec{x})$ in L_{PB} . Let $r \geq 0 \in \mathbb{Q}$. The following formula is in L_{PB} : $\exists y(|y| \leq r \wedge \phi(y, \vec{x}))$.
7. Consider a formula $\phi(y, \vec{x})$ in L_{PB} . Let $r \geq 0 \in \mathbb{Q}$. The following formula is in L_{PB} : $\forall y(|y| \leq r \Rightarrow \phi(y, \vec{x}))$ ■

Note: For real valued relations $\mathcal{R}_1(\vec{x})$ and $\mathcal{R}_2(\vec{x})$, we will write $\mathcal{R}_1(\vec{x}) \leq \mathcal{R}_2(\vec{x})$ in L_{PB} to abbreviate the formula $\bigwedge_{q \in \mathbb{Q}} \mathcal{R}_1(\vec{x}) \leq q \vee \mathcal{R}_2(\vec{x}) \geq q$. In a similar manner we will abbreviate $\mathcal{R}_1(\vec{x}) \geq \mathcal{R}_2(\vec{x})$ and $\mathcal{R}_1(\vec{x}) = \mathcal{R}_2(\vec{x})$.

We now recall the definition of approximate formulas for L_{PB} (see [4] for more details).

Definition 2.7. Definition of \models_{AP} for L_{PB} .

For every $\phi \in L_{PB}$ and every integer n , define ϕ_n as follows:

- $(|t| \leq r)_n : |t| \leq r + 1/n$ and $(|t| \geq r)_n : |t| \geq r - 1/n$. Likewise, $(\mathcal{R}(\vec{t}) \leq r)_n : \mathcal{R}(\vec{t}) \leq r + 1/n$ and $(\mathcal{R}(\vec{t}) \geq r)_n : \mathcal{R}(\vec{t}) \geq r - 1/n$.
- $(\phi \wedge \psi)_n : \phi_n \wedge \psi_n$ and $(\phi \vee \psi)_n : \phi_n \vee \psi_n$.
- $(\bigwedge_{i=1}^\infty \phi_i)_n : \bigwedge_{i=1}^n (\phi_i)_n$.
- $(\exists y(|y| \leq r \wedge \phi(\vec{x}, y)))_n : \exists y(|y| \leq r + 1/n \wedge \phi_n(\vec{x}, y))$.
- $(\forall y(|y| \leq \vec{r} \Rightarrow \phi(\vec{x}, y)))_n : \forall y(|y| \leq r - 1/n \Rightarrow \phi_n(\vec{x}, y))$.

Finally, for $\phi \in L_{PB}$, $\mathbf{E} \models_{AP} \phi(\vec{a})$ iff $\mathbf{E} \models \bigwedge_{n=1}^\infty \phi_n(\vec{a})$. ■

Notation: to avoid long formulas, we will abbreviate

$$\exists x_1(\|x_1\| \leq r_1 \wedge \exists x_2(\|x_2\| \leq r_2 \wedge \dots \exists x_s(\|x_s\| \leq r_s \wedge \phi) \dots)$$

by $\exists \vec{x}(\bigwedge_{i=1}^s \|x_i\| \leq r_i \wedge \phi)$.

It is easy to see ([4]) that the following is true:

Theorem 2.8. *Let E be a normed space structure, and let $\phi(\vec{x})$ be a positive bounded formula. The following holds for every normed structure E :*

- For every integer n , $E \models \phi_{n+1}(\vec{a}) \Rightarrow \phi_n(\vec{a})$;
- If $E \models \phi(\vec{a})$ then $E \models_{AP} \phi(\vec{a})$.

Although L_{PB} does not have the negation connective, one can define a weak approximate negation operator in L_{PB} inspired by Henson's weak negation operator ([5]).

Definition 2.9. Weak approximate negation operator.

Fix a signature Ω . For every integer n and every formula $\phi \in L_{PB}$ we define the operator $neg(\phi, n)$ as follows:

1. If t is a term of the first order language corresponding to Ω and r is a rational number, then $neg(\|t\| \leq r, n) : \|t\| \geq r + 1/n$ and $neg(\|t\| \geq r, n) : \|t\| \leq r - (1/n)$. Likewise, $neg(\mathcal{R}(\vec{t}) \leq r, n) : \mathcal{R}(\vec{t}) \geq r + 1/n$ and $neg(\mathcal{R}(\vec{t}) \geq r, n) : \mathcal{R}(\vec{t}) \leq r - 1/n$.
2. $neg(\phi \wedge \psi, n) : neg(\phi, n) \vee neg(\psi, n)$.
3. $neg(\phi \vee \psi, n) : neg(\phi, n) \wedge neg(\psi, n)$.
4. $neg(\bigwedge_{i=1}^{\infty} \phi_i, n) : \bigvee_{i=1}^n neg(\phi_i, n)$.
5. $neg(\exists x(\|x\| \leq r \wedge \phi, n) : \forall x(\|x\| \leq r + (1/n) \Rightarrow neg(\phi, n))$.

6. $neg(\forall x(\|x\| \leq r \Rightarrow \phi, n) : \exists x(\|x\| \leq \vec{r} - (1/n) \wedge neg(\phi, n)). \blacksquare$

The main property of the weak approximate negation operator is given by the following lemma. We call the subcollection of L_{PB} containing the atomic formulas and closed under finite conjunction, disjunction, and the existential and universal bounded quantification the **finitary part of L_{PB}** .

Lemma 2.10. *For every formula $\phi \in L_{PB}$,*

- $\forall n \in \omega$, $neg(\phi, n)$ is in the finitary part of L_{PB} .
- For every structure E and every $\vec{a} \in E$,

$$E \not\models_{AP} \phi(\vec{a}) \text{ iff } \exists m \in \omega \ E \models neg(\phi, m).$$

- For every integer n , for every structure E and every \vec{a} in E ,

$$E \models \neg(\phi_n) \Rightarrow neg(\phi, n + 1).$$

Proof. The proof is direct and is left to the reader. \square

We now define the fully infinitary logic L_A based on L_{PB} .

Definition 2.11. Definition of L_A .

Fix a signature Ω . We define L_A by induction in formulas:

1. $L_{PB} \subset L_A$.
2. If $\phi_1, \phi_2, \dots, \phi_i \dots$ ($i < \omega$) is a collection of formulas in L_A , then for every integer n , $\bigwedge_{i=1}^n \phi_i$, and $\bigwedge_{i=1}^\infty \phi_i$ are formulas in L_A .
3. If ϕ is a formula in L_A then $\neg\phi$ is also a formula in L_A .

4. Consider a formula $\phi(y_1, \dots, y_n, \dots, \vec{x})$ in L_A . Let $\vec{r} = (r_1, \dots, r_n, \dots)$ be a corresponding vector of rational numbers. The following formula is in L_A :

$$\exists(y_1, \dots, y_n, \dots) \left(\bigwedge_{n=1}^{\infty} \|y_n\| \leq r_n \wedge \phi(y_1, \dots, y_n, \dots, \vec{x}) \right). \blacksquare$$

Notation: to avoid very long formulas we will abbreviate

$$\exists(y_1, \dots, y_n, \dots) \left(\bigwedge_{n=1}^{\infty} \|y_n\| \leq r_n \wedge \phi(\vec{x}, \vec{y}) \right)$$

by $\exists \vec{y}(\|\vec{y}\| \leq \vec{r} \wedge \phi(\vec{x}, \vec{y}))$. Likewise $\neg \exists \vec{y}(\|\vec{y}\| \leq \vec{r} \wedge \phi(\vec{y}, \vec{x}))$ will be abbreviated by $\forall \vec{y}(\|\vec{y}\| \leq \vec{r} \Rightarrow \neg \phi(\vec{y}, \vec{x}))$. We will also abbreviate $\neg \bigwedge \neg$ by \bigvee and $\neg(\phi \wedge \neg \psi)$ by $\phi \Rightarrow \psi$.

Finally, given a countable set $A = \{a_1, \dots, a_n, \dots\}$ with a fixed enumeration and countable formulas $\{\phi_a\}_{a \in A}$ we understand by $\bigwedge_{a \in A} \phi_a$ the formula $\bigwedge_{n=1}^{\infty} \phi_{a_n}$. Likewise, for an arbitrary integer m , we understand by $\bigwedge_{a \in A \upharpoonright m} \phi_a$ the formula $\bigwedge_{n=1}^m \phi_{a_n}$.

The notion of satisfaction ($\mathbf{E} \models \phi(\vec{a})$) for Ω -structures \mathbf{E} , with \vec{a} a vector of elements in E and for $\phi \in L_A$ is the natural one and we are not going to do it here. The interested reader is directed to [5] for more details.

Example 2.12. Expressive Power of L_A .

We show that the property of reflexivity can be expressed in the logic L_A .

For any Banach space $(X, \|\cdot\|)$, let B_1 denote the unitary ball. A characterization of reflexivity due to James ([8]) (see also [14]) that does not require any mention of the dual is the following:

A Banach space $(X, \|\cdot\|)$ is reflexive iff

$$\forall \epsilon > 0 \forall \{x_i\}_{i=1}^{\infty} \subseteq B_1 \exists k \in \omega \text{ dist}[\text{conv}(\{x_1, \dots, x_k\}), \text{conv}(\{x_{k+1}, \dots\})] \leq \epsilon$$

Here, for any set $A \subseteq X$, $conv(A)$ is the convex hull spawned by A . Similarly, given two sets $A, B \in X$, $dist[A, B] = inf\{\|x - y\| : x \in A, y \in B\}$.

Let Ω be the empty signature. Then the Ω -structures are the normed spaces. The following sentence of L_A expresses reflexivity for the closure of these structures.

$$\bigwedge_{n=1}^{\infty} \forall \vec{x} (\|\vec{x}\| \leq 1 \Rightarrow \bigvee_{k=1}^{\infty} \bigvee_{r=1}^{\infty} \bigvee_{\vec{a} \in CO(k)} \bigvee_{\vec{b} \in CO(r)} \|\sum_{i=1}^k a_i x_i - \sum_{j=1}^r b_j x_{k+j}\| < (1/n))$$

Here, $\forall s \in \omega$, $CO(s)$ is the subset of \mathbb{Q}^s made of all the s -tuples (a_1, \dots, a_s) such that $\sum_{i=1}^s a_i = 1$ and $a_1, \dots, a_s \geq 0$. ■

3 Approximate formulas for L_A

Our intention is to generate approximations of all the formulas in L_A by using the formulas in L_{PB} as building blocks. As mentioned in the introduction, the main problem arises from the negation connective. We will use the weak approximate negation operator ($neg(., ., .)$) defined in the previous section to solve this problem.

Formally, we associate to every formula ϕ in L_A a set of indices $I(\phi)$ (the branches of the tree of approximate formulas) and for every $h \in I(\phi)$ a formula $[\phi]_h \in L_{PB}$. Intuitively, for every branch $h \in I(\phi)$, the approximate formulas of $[\phi]_h$ (the collection $\{([\phi]_h)_n | n \in \omega\}$) are going to “approach” ϕ as n tends to ∞ .

The notions of $I(\phi)$ and $([\phi]_h)_n$ were introduced (in a different presentation) in [12].

Notation: Given two formulas ϕ, σ , we will write $\phi \equiv \sigma$ if ϕ and σ are identical formulas.

In the rest of this section we fix a signature Ω .

Definition 3.1. Approximate formulas in L_A .

For any formula $\phi(\vec{x})$ in L_A we define by induction in formulas:

- a set $I(\phi)$ of branches;
- $\forall h \in I(\phi)$, formulas $[\phi]_h \in L_{PB}$.

Formulas in L_{PB} . $\forall \phi \in L_{PB}$, $I(\phi) = \{\emptyset\}$. Furthermore, $[\phi]_\emptyset : \phi$.

Countable (Finite) Conjunction. For any countable (or finite) collection $\{\phi_i\}_{i=1}^\infty$ ($\{\phi_i\}_{i=1}^m$) of formulas in L_A , we define:

- $I(\bigwedge_{i=1}^\infty \phi_i(\vec{x})) = \prod_{i=1}^\infty I(\phi_i)$ (the cartesian product of the $I(\phi_i)$) (or $I(\bigwedge_{i=1}^m \phi_i(\vec{x})) = \prod_{i=1}^m I(\phi_i)$).
- For every h in $I(\bigwedge_{i=1}^\infty \phi_i)$,

$$[\bigwedge_{i=1}^\infty \phi_i]_h : \bigwedge_{i=1}^\infty [\phi_i]_{h(i)} \text{ (or } [\bigwedge_{i=1}^m \phi_i]_h : \bigwedge_{i=1}^m [\phi_i]_{h(i)})$$

Negation. For any formula ϕ in L_A , we have:

- $I(\neg\phi) \subseteq (I(\phi) \times \omega)^\omega$ is the collection of all maps $f = (f_1, f_2)$ with the following “weak” surjectivity property:

$$\forall h \in I(\phi) \exists s \in \omega, ([\phi]_h)_{f_2(s)} \equiv ([\phi]_{f_1(s)})_{f_2(s)}$$

- For every $f = (f_1, f_2) \in I(\neg\phi)$, $[\neg\phi]_f : \bigwedge_{s=1}^\infty \text{neg}([\phi]_{f_1(s)}, f_2(s))$

Existential. For every formula $\phi(\vec{y}, \vec{x})$, for every corresponding vector \vec{r} of rational numbers, we have:

- $I(\exists \vec{y} (||\vec{y}|| \leq \vec{r} \wedge \phi(\vec{y}, \vec{x}))) = I(\phi(\vec{y}, \vec{x}))$.

- For every h in $I(\exists \vec{y}(\|\vec{y}\| \leq \vec{r} \wedge \phi(\vec{y}, \vec{x})))$, let $Ind(n)$ be the value of the maximal index such that x_{In} appears free in $([\phi(\vec{y}, \vec{x})]_h)_n$. We define

$$[\exists \vec{y}(\|\vec{y}\| \leq \vec{r} \wedge \phi(\vec{y}, \vec{x}))]_h : \bigwedge_{n=1}^{\infty} \exists \vec{y} \left(\bigwedge_{s=1}^{Ind(n)} \|y_s\| \leq r_s \wedge ([\phi(\vec{y}, \vec{x})]_h)_n \right). \blacksquare$$

The formulas $([\phi]_h)_n$ are the **approximate formulas** of ϕ .

Definition 3.2. Approximate Truth.

Fix an Ω -structure \mathbf{E} . Let $\phi(\vec{x})$ be an arbitrary formula in L_A . We say that $\mathbf{E} \models_{AP} \phi(\vec{a})$ (\mathbf{E} approximately satisfies ϕ) iff

$$\exists h \in I(\phi(\vec{x})) \forall n \in \omega, \mathbf{E} \models ([\phi(\vec{a})]_h)_n \blacksquare$$

Note It is clear from the above definition that \models_{AP} “a la Henson” and \models_{AP} for L_A coincide for formulas in L_{PB} . Hence, from now on, there shall be no confusion concerning the notion of \models_{AP} being used.

4 Uniformity Theorem for L_A

In this section we fix a countable signature Ω .

We begin by proving that Henson’s compactness theorem for \models_{AP} in L_{PB} in fact holds for \models_{AP} in L_A . This is not surprising, since the approximate formulas in L_A are positive bounded formulas.

Let us recall first three fundamental results for approximate truth in L_{PB} . The interested reader can get details of the proofs in [5] or [7].

Theorem 4.1. *Henson’s Compactness Theorem*

Let Σ be a theory in L_{PB} , such that for every finite $F = \{\sigma_i : i \leq k\} \subseteq \Sigma$, for every integer n there exists a normed space structure E_n such that

$E_n \models \bigwedge_{i=1}^k (\sigma_i)_n$. Then there exists a normed space structure E such that $E \models_{AP} \Sigma$.

For the next theorem we need a definition.

Definition 4.2. κ -saturated normed structures.

A normed space structure E is κ -saturated if it approximately realizes any consistent set of formulas in L_{PB} containing less than κ constants and norm bounds for elements from E . ■

Theorem 4.3. \aleph_1 -saturated structures.

For any normed structure E , there exists an approximate elementary extension F of E (i.e. E and F approximately satisfy the same formulas in L_{PB} with parameters in E) that is \aleph_1 -saturated.

The final theorem shows that \aleph_1 -saturated structures are "rich" for L_{PB} :

Theorem 4.4. If E is an \aleph_1 -saturated and $\phi(\vec{x}) \in L_{PB}$ then $E \models \phi(\vec{a})$ iff $E \models_{AP} \phi(\vec{a})$.

We use the above theorems to prove first that Henson's Compactness Theorem holds in fact for \models_{AP} in L_A .

Definition 4.5. Let Θ be a collection of sentences in L_A . We say that Θ is approximately finite consistent iff there exists a set of branches $\Lambda = \{h(\sigma) \in I(\sigma) : \sigma \in \Theta\}$ such that for every finite subset $F \subset \Theta$, for every integer n , there exists a normed space structure \mathbf{E}_n such that:

$$\forall \sigma \in F, \mathbf{E}_n \models ([\sigma]_{h(\sigma)})_n. \blacksquare$$

Note that the above definition of approximate finite consistency for L_A restricted to L_{PB} coincide with Henson's definition of finite consistency for L_{PB} ([5]).

The main consequence of the previous lemmas is the following Model Existence Theorem for (L_A, \models_{AP}) and (L_A, \models) .

Theorem 4.6. *Fix an approximately finite consistent collection Θ of sentences in L_A . Then there exists a normed space structure \mathbf{E} such that $E \models_{AP} \Theta$, and $E \models \Theta$.*

Proof. Fix Θ as in the hypothesis of the theorem, and let $\Lambda = \{h(\sigma) \in I(\sigma) : \sigma \in \Theta\}$ be the set of branches associated with Θ . Let $T = \{[\sigma]_{h(\sigma)} : \sigma \in \Theta\}$. Clearly T is a theory in L_{PB} that satisfies the hypothesis of Henson's Compactness Theorem. It follows from Theorem 4.3 that there exists an \aleph_1 -saturated structure E such that: $E \models_{AP} T$. By the definition of \models_{AP} in L_A it follows that $E \models_{AP} \Theta$. It remains to prove that $E \models \Theta$.

We claim:

For every formula $\phi(\vec{x}) \in L_A$, $E \models_{AP} \phi(\vec{a})$ iff $E \models \phi(\vec{a})$.

proof: By induction in the formulas of L_A . The proofs for L_{PB} and for the countable (or finite) conjunction steps are direct, and are left to the reader.

Negation. \Rightarrow . Assume that $E \models_{AP} \neg \phi(\vec{a})$. Assume also, in order to get a contradiction, that $E \models \phi(\vec{a})$. By induction hypothesis it follows that $E \models_{AP} \phi(\vec{a})$ which implies that there exists a branch $h \in I(\phi)$ such that

$$E \models \bigwedge_{n=1}^{\infty} ([\phi]_h)_n. \quad (1)$$

However, since $E \models_{AP} \neg \phi(\vec{a})$, it follows from the definition of the approximate truth for the negation (Definition 3.1) that there exists a function $f = (f_1, f_2) : \omega \rightarrow I(\phi) \times \omega$ with the following "weak" surjectivity property:

$$\forall g \in I(\phi) \exists s, ([\phi]_g)_{f_2(s)} \equiv ([\phi]_{f_1(s)})_{f_2(s)}$$

and such that $E \models \bigwedge_{s=1}^{\infty} \text{neg}([\phi(\vec{a})]_{f_1(s)}, f_2(s))$. From these two properties of f it follows that there exists an m such that $E \models \text{neg}([\phi(\vec{a})]_h, m)$, which implies from the properties of the weak approximate negation (Lemma 2.10) that $E \not\models \bigwedge_{n=1}^{\infty} ([\phi]_h)_n$, but this contradicts the statement 1.

\Leftarrow . Assume that $E \models \neg\phi(\vec{a})$. By induction hypothesis, we get that $E \not\models_{AP} \phi(\vec{a})$.

Hence, for every branch $h \in I(\psi)$, there exists an integer m such that $E \not\models ([\phi(\vec{a})]_h)_m$. We invoke now Lemma 2.10 to obtain that there exists an integer n such that $E \models \text{neg}([\phi(\vec{a})]_h, n)$.

Consider now that the collection of all formulas of the form $\text{neg}([\phi(\vec{a})]_h, n)$ that hold in E . From Lemma 2.10 it follows that those formulas are finitary and belong to L_{PB} so they are at most countable. This implies that we can construct a function $f = (f_1, f_2) : \omega \rightarrow I(\phi) \times \omega$ with the "weak" surjectivity property (i.e. $\forall g \in I(\phi) \exists s ([\phi]_g)_{f_2(s)} \equiv ([\phi]_{f_1(s)})_{f_2(s)}$) and such that $E \models \bigwedge_{s=1}^{\infty} \text{neg}([\phi(\vec{a})]_{f_1(s)}, f_2(s))$.

We get then from the definition of approximate formulas for L_A (Definition 3.1) that $E \models_{AP} \neg\phi(\vec{a})$. This completes the proof of the negation step.

Existential . There is only one interesting direction. Assume that $E \models_{AP} \exists \vec{x} (||\vec{x}|| \leq \vec{r} \wedge \phi(\vec{a}, \vec{x}))$. Then there exists $h \in I(\phi)$ such that

$$E \models \bigwedge_{n=1}^{\infty} \exists \vec{x} \left(\bigwedge_{i=1}^{Ind(n)} ||x_i|| \leq r_i \wedge ([\phi(\vec{a}, \vec{x})]_h)_n \right).$$

Since the above formula is a countable conjunction of finitary formulas in L_{PB} and E is \aleph_1 -saturated, it follows that the conjunction $\bigwedge_{i=1}^{\infty} ||x_i|| \leq r_i \wedge [\phi(\vec{a}, \vec{x})]_h$ is approximately realized in E for some \vec{b} . This implies that $E \models_{AP} \phi(\vec{a}, \vec{b})$, and hence, by induction hypothesis, $E \models \exists \vec{x} (||\vec{x}|| \leq$

$\vec{r} \wedge \phi(\vec{a}, \vec{x})$. This completes the proof of the existential step and of the claim.

From the above claim it follows that $E \models \Theta$. This completes the proof of the theorem. \square

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We are now ready to prove the main result of the paper: the Uniformity Theorem.

Theorem 4.7. *Uniformity Theorem for L_A .*

Let Σ be a theory in L_{PB} . Let $\neg\phi$ be a sentence in L_A . If $\Sigma \models \neg\phi$ then for every $h \in I(\phi)$ there exists an integer n such that $\Sigma \models \neg([\phi]_h)_n$.

Proof. Assume, in order to get a contradiction, that there exists $h \in I(\phi)$ such that for every integer n there is a normed structure E_n satisfying:

$$E_n \models \Sigma \text{ and } E_n \models ([\phi]_h)_n$$

We can now invoke Theorem 2.8 to obtain that there exists a normed structure E such that $E \models \Sigma$ and $E \models \phi$, but this is a contradiction with the hypothesis. \square

5 Applications

A first corollary of the Uniformity Theorem concerns sentences of the form $\sigma \Rightarrow \theta$ where $\sigma, \theta \in L_{PB}$.

Corollary 5.1. *"Almost" Versions.*

Fix a signature Ω . Suppose that Σ is a theory in L_{PB} such that $\Sigma \models (\sigma \Rightarrow \theta)$, for $\sigma, \theta \in L_{PB}$. Then for every integer n there exists an integer m

such that:

$$\Sigma \models \sigma_m \Rightarrow \theta_n$$

Proof. It is enough to decode the approximate formulas corresponding to

$$\sigma \Rightarrow \theta : \neg(\sigma \wedge \neg\theta)$$

Since $\sigma, \theta \in L_{PB}$, then $I(\sigma) = \{\emptyset\}$ and $I(\neg\theta) = (\{\emptyset\} \times \omega)^\omega$. It follows that $I(\sigma \wedge \neg\theta) = \{\emptyset\} \times (\{\emptyset\} \times \omega)^\omega$.

Note that for every integer n , the constant function $[n+1] : \omega \rightarrow \{\emptyset\} \times \{n+1\}$ belongs to $I(\neg\theta)$. Hence it follows from the Uniformity Theorem (Theorem 4.7) that there exists an integer m such that:

$$\Sigma \models \neg(\sigma_m \wedge ([\neg\theta]_{[n+1]})_m)$$

which decoded says

$$\Sigma \models \neg(\sigma_m \wedge \bigwedge_{i=1}^m (neg(\theta, [n+1](i)))_m)$$

which is

$$\Sigma \models \sigma_m \Rightarrow \neg((neg(\theta, n+1))_m).$$

Invoking now Lemma 2.10 we get that $\Sigma \models \sigma_m \Rightarrow \theta_n$. □

Example 5.2. Ulam's Theorem

Note first that Ulam's Theorem is equivalent to the version where the target and domain Banach space are the same.

Fix now an arbitrary integer k and let Ω_k be the signature induced by any normed space structure $\mathbf{E} = (E, T)$ where $T : E \rightarrow E$ is a continuous map with the property that $\forall x \in E \ ||T(x)|| \leq k||x||$.

Consider, in Ω_k , the theory

$$\Sigma_k = \{\forall x(|x| \leq n \Rightarrow \|T(x)\| \leq k|x|) : n \in \omega\} \wedge$$

$$\{\|T(0)\| = 0\} \wedge$$

$$\{\forall x(|x| \leq n \Rightarrow \exists y(|y| \leq kn \wedge \|T(x) - y\| = 0)) : n \in \omega\}$$

that says that T sends 0 to 0 and is an onto map. Clearly $\Sigma_k \subseteq L_{PB}$.

Note that the version of Ulam's Theorem mentioned above can be expressed as a formula in L_A :

$$\phi : (\forall x, y(|x|, |y| \leq 1 \Rightarrow \|T(x) - T(y)\| = \|x - y\|))$$

$$\Rightarrow (\forall x, y(|x|, |y| \leq 1 \Rightarrow \|T(x) + T(y) - x - y\| = 0))$$

which is of the form

$$\phi : (\sigma \Rightarrow \theta) : \neg(\sigma \wedge \neg\theta)$$

with $\sigma, \theta \in L_{PB}$. Since $\Sigma_k \models \phi$ we can invoke the Corollary 5.1 for formulas based on the signature Ω_k to obtain that for every integer n there exists an integer m such that:

$$\Sigma_k \models [\forall x, y(|x|, |y| \leq 1 - 1/m \Rightarrow$$

$$\|x - y\| - 1/m \leq \|T(x) - T(y)\| \leq \|x - y\| + 1/m)] \Rightarrow$$

$$[\forall x, y(|x|, |y| \leq 1 - 1/n \Rightarrow \|T(x) + T(y) - x - y\| \leq 1/n)]$$

which easily implies Gervitz's version. ■

Example 5.3. Behrends' Theorem.

Fix an integer k and two rational numbers $1 \leq p, q \leq \infty$ and let Ω_k be the signature induced by a normed space structure $\mathbf{E} = (E, P, Q, e_1, e_2, f^p, f^q)$ with:

- P, Q being linear projections (i.e. $P^2 = P$ and $Q^2 = Q$) with the property that for every $x \in E$, $\|P(x)\| \leq k\|x\|$ and $\|Q(x)\| \leq k\|x\|$,
- e_1, e_2 being vectors with norm 1,
- f^p being a real valued function satisfying the same modulus of continuity and the same bounds as the function $\|\cdot\|^p$,
- f^q being a real valued function satisfying the same modulus of continuity and the same bounds as the function $\|\cdot\|^q$.

Consider, in Ω_k , the theory $\Sigma_k =$

$$\{\forall x, y (\|x\|, \|y\| \leq r \Rightarrow \|P(ax + by) - aP(x) - bP(y)\| = 0) : a, b, r \in \mathbb{Q}\} \cup$$

$$\{\forall x, y (\|x\|, \|y\| \leq r \Rightarrow \|Q(ax + by) - aQ(x) - bQ(y)\| = 0) : a, b, r \in \mathbb{Q}\} \cup$$

$$\{\forall x (\|x\| \leq r \Rightarrow \|P(P(x)) - x\| = 0) : r \in \mathbb{Q}\} \cup$$

$$\{\forall x (\|x\| \leq r \Rightarrow \|Q(Q(x)) - x\| = 0) : r \in \mathbb{Q}\} \cup$$

$$\{\forall x (\|x\| \leq r \Rightarrow (\|x\| \leq s \vee s^p \leq f^p(x) \leq r^p)) : s < r \text{ in } \mathbb{Q}\} \cup$$

$$\{\forall x (\|x\| \leq r \Rightarrow (\|x\| \leq s \vee s^q \leq f^q(x) \leq r^q)) : s < r \text{ in } \mathbb{Q}\} \cup$$

$$\{\|e_1\| = 1 \wedge \|e_2\| = 1 \wedge \|e_1 - e_2\| \geq 1/2\}.$$

The last sentence listed for Σ_k is equivalent (using Riesz's Lemma) to the statement that the dimension is > 2 . Clearly, $\Sigma_k \subseteq L_{PB}$.

Behrends' Theorem can be expressed by the following sentence in L_A :

$$\begin{aligned} & [\forall x (\|x\| \leq 1 \Rightarrow (f^p(P(x)) + f^p(x - P(x))) = f^p(x)) \wedge \forall x (\|x\| \leq 1 \\ & \quad \Rightarrow (f^q(P(x)) + f^q(x - Q(x))) = f^q(x))] \\ & \Rightarrow [\forall x (\|x\| \leq 1 \Rightarrow \|P(Q(x)) - Q(P(x))\| = 0 \wedge \|f^p(x) - f^q(x)\| = 0)] \end{aligned}$$

It follows then from Corollary 5.1 that for every integer n there exists an integer m such that for every normed space E with $\dim > 2$,

$$\begin{aligned}
& [\forall x(\|x\| \leq 1 - 1/m \Rightarrow \\
& (f^p(x) - 1/m \leq f^p(P(x)) + f^p(x - P(x))) \leq f^p(x) + 1/m) \wedge \\
& \forall x(\|x\| \leq 1 \\
& \Rightarrow (f^q(x) - 1/m \leq f^q(P(x)) + f^q(x - Q(x))) \leq f^q(x) + 1/m)] \\
& \Rightarrow [\forall x(\|x\| \leq 1 - 1/n \Rightarrow \|P(Q(x)) - Q(P(x))\| \leq 1/n \wedge \|f^p(x) - f^q(x)\| \leq 1/n)].
\end{aligned}$$

from which one obtains the result from Cambern, Jaroz and Wodinski. ■

The last example concerns a different type of formula: an infinitary one.

Example 5.4. Krivine's Theorem.

A celebrated result by Krivine ([10]) states that for every basic Schrauder sequence of unitary vectors $\{x_n\}_{n=1}^\infty$ in a normed space E , there exists a $p \in [1, \infty)$ such that the usual basis of one of the spaces ℓ_p (or c_0) is block finitely representable in $\{x_n\}_{n=1}^\infty$. We refer the reader to [11] for the definition and properties of the basic Schrauder sequences as well as of the block finitely representable basis.

Let Ω be the empty signature. Let $\mathbb{Q}^\# = (\mathbb{Q} \cap [1, \infty])$. For a fix integer n , let \vec{q} denote a vector $(q_1, q_2, \dots, q_{n+1})$ of integers such that $q_1 < q_2 < \dots < q_{n+1}$ and let $V_n \subset \omega^{n+1}$ be the collection of all such vectors.

A weaker version of this theorem has the form: For every $K \geq 1$, for every $\epsilon > 0$, for every $n \in \omega$, for every normed space E ,

$$\begin{aligned}
& E \models \forall \vec{x}(\|\vec{x}\| \leq 1 \Rightarrow \\
& [BaseK(\vec{x}) \Rightarrow \bigvee_{p \in \mathbb{Q}^\#} \bigvee_{\vec{q} \in V_n} \bigvee_{\vec{b} \in Q^{q_{n+1}}} \theta^{n,p,\epsilon}(\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i)]
\end{aligned}$$

where

•

$$BaseK(\vec{x}) : \bigwedge_{i=1}^{\infty} \|x_i\| = 1 \wedge \bigwedge_{n,m \in \omega} \bigwedge_{\vec{a} \in Q^{n+m}} \left\| \sum_{i=1}^n a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n+m} a_i x_i \right\|$$

is a positive formula that states that the sequence $\{x_i\}_{i=1}^{\infty}$ is a basic Schrauder sequence with constant K ;

• for every integers n and $p \in Q^{\#}$,

$$\theta^{n,p,\epsilon}(y_1, \dots, y_n) : \bigwedge_{\vec{c} \in Q^n} \sqrt[p]{\sum_{i=1}^n |c_i|^p} \leq \left\| \sum_{j=1}^n c_j y_j \right\| \leq (1 + \epsilon) \sqrt[p]{\sum_{i=1}^n |c_i|^p}$$

states that the usual basis of the space ℓ_p^n is $(1 + \epsilon)$ -equivalent to the vectors (y_1, \dots, y_n) .

Note that the weaker form of Krivine's Theorem can be written as: For every $K \geq 1$, for every $\epsilon > 0$, for every integer n ,

$$\models \neg \exists \vec{x} (\|\vec{x}\| \leq 1 \wedge$$

$$[BaseK(\vec{x}) \wedge \bigwedge_{p \in Q^{\#}} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{qn+1}} \neg \theta^{n,p,\epsilon}(\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i)]$$

Note also that, by virtue of the finite dimensionality of the ℓ_p^n , for every integer n , for every $p \in Q^{\#}$, for every $\epsilon > 0$ there exists an integer $w(n, p, \epsilon)$ such that

$$\models (\theta^{n,p,\epsilon})_{w(n,p,\epsilon)} \Rightarrow \theta^{n,p,2\epsilon}$$

Using the above remark, define for any $\epsilon \geq 0$, for every integer n , for every $K \geq 1$, the function

$$h : Q^{\#} \times V_n \times Q^{qn+1} \rightarrow \omega$$

such that $h(p, (q_1, q_2, \dots, q_{n+1}), \vec{b}) = w(n, p, \epsilon) + 1$. We leave to the reader the verification that the pair (\emptyset, h) is a branch of the tree of approximations of the formula

$$BaseK(\vec{x}) \wedge \bigwedge_{p \in \mathbb{Q}^\#} \bigwedge_{\vec{q} \in V_n} \bigwedge_{\vec{b} \in Q^{q_{n+1}}} \neg \theta^{n,p,\epsilon} \left(\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i \right)$$

For every $K \geq 1$, for every $\epsilon > 0$, for every integer n we can invoke the Uniformity Theorem for the branch (\emptyset, h) to obtain that there exists an integer r such that:

$$\models \neg \exists \vec{x} (||\vec{x}|| \leq 1 + (1/r) \wedge [(BaseK(\vec{x}))_r \wedge \bigwedge_{p \in \mathbb{Q}^\# \uparrow r} \bigwedge_{\vec{q} \in V_n \uparrow r} \bigwedge_{\vec{b} \in Q^{q_{n+1}} \uparrow r} neg(\theta^{n,p,\epsilon}(\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i), h(p, \vec{q}, \vec{b}))])]$$

which implies, using again Lemma 2.10, the property of $w(n, p, \epsilon)$ and the definition of h , that

$$\models \forall \vec{x} (||\vec{x}|| \leq 1 \Rightarrow [(BaseK(\vec{x}))_r \Rightarrow \bigvee_{p \in \mathbb{Q}^\# \uparrow r} \bigvee_{\vec{q} \in V_n \uparrow r} \bigvee_{\vec{b} \in Q^{q_{n+1}} \uparrow r} \theta^{n,p,2\epsilon}(\sum_{i=q_1+1}^{q_2} b_i x_i, \sum_{i=q_2+1}^{q_3} b_i x_i, \dots, \sum_{i=q_n+1}^{q_{n+1}} b_i x_i)]).$$

This last statement can be written as follows:

For every $K > 1$, for every $\epsilon > 0$, for every integer n , there exists a finite collection $I = \{p_1, \dots, p_r\} \in \mathbb{Q} \cap [1, \infty]$ and an integer m such that for every finite basic sequence $(x_i)_{i=1}^m$ with basic constant K in any normed space, there exists a $p \in I$ and a block basic sequence that is $1 + \epsilon$ equivalent to the usual basis of ℓ_p^n .

Compare this result with the Uniform Version of Krivine's Theorem obtained by Rosenthal ([13]):

Fix arbitrary $K \geq 1$, $n \in \omega$ and $\epsilon > 0$. There exists an m such that if $(x_i)_{i=1}^m$ is a finite basic sequence in any Banach space with basis constant K , then there exists $1 \leq p \leq \infty$ and a block sequence $(y_i)_{i=1}^n$ so that $(y_i)_{i=1}^n$ is $(1 + \epsilon)$ -isomorphic to the unit vector basis of ℓ_p^n . ■

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